ELSEVIER

International Journal of Solids and Structures 41 (2004) 847–857

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A continuum model for size-dependent deformation of elastic films of nano-scale thickness

L.H. He [a], C.W. Lim [b,*], B.S. Wu [c]

[a] *Key Laboratory of Mechanical Behavior and Design of Materials, CAS, University of Science and Technology of China, Hefei, Anhui 230026, PR China*

[b] *Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon 0000, Hong Kong*

[c] *Department of Mathematics, Jilin University, Changchun, Jilin 130023, PR China*

Received 21 May 2003; received in revised form 24 September 2003

Abstract

Ultra-thin elastic films of nano-scale thickness with an arbitrary geometry and edge boundary conditions are analyzed. An analytical model is proposed to study the size-dependent mechanical response of the film based on continuum surface elasticity. By using the transfer-matrix method along with an asymptotic expansion technique of small parameter, closed-form solutions for the mechanical field in the film is presented in terms of the displacements on the mid-plane. The asymptotic expansion terminates after a few terms and exact solutions are obtained. The mid-plane displacements are governed by three two-dimensional equations, and the associated edge boundary conditions can be prescribed on average. Solving the two-dimensional boundary value problem yields the three-dimensional response of the film. The solution is exact throughout the interior of the film with the exception of a thin boundary layer having an order of thickness as the film in accordance with the Saint-Venant's principle.

Keywords: Thin film; Surface elasticity; Size effect

1. Introduction

The surface of a solid is a region with small thickness which has its own atom arrangement and property differing from the bulk. For a solid with a large size, the surface effects can be ignored because the volume ratio of the surface region to the bulk is very small. However, for small solids with large surface-to-bulk ratio the significance of surfaces is likely to be important. This is extremely true for nano-scale materials or structures. Recently, mechanical experiments of nano-scale bars (Wong et al., 1997) and plates (Rose et al., 2000) indicate that the effective elastic properties of these minute structural elements strongly depend on their size. The understanding and modeling of such a size-dependent phenomenon has become an active subject of much research (Sheehan and Lieber, 1996; Yakobson and Smalley, 1997; Terrones et al., 1999).

* Corresponding author. Tel.: +852-2788-7285; fax: +852-2788-7612.
E-mail address: bccwlim@cityu.edu.kh (C.W. Lim).

doi:10.1016/j.ijsolstr.2003.10.001

Classical elasticity lacks an intrinsic length scale, and thus cannot be used to model the size effect. Atomistic simulation (Sun and Zhang, 2003), though very powerful in pursuing the details at microscopic level, seems too complex for practical applications as it needs tremendous computation. An efficient approach has been developed by Miller and Shenoy (2000) upon the continuum concept of surface stress. They examined unidirectional tension and pure bending of nano-scale bars and plates, and found more remarkable size effect in bending than in tension. The results are in excellent agreement with their atomistic simulation by embedded atom method for face-centered cubic aluminum and the Stillinger–Weber model for silicon. The present paper will seek to extend this continuum approach, and it is devoted to analyzing the static response of ultra-thin elastic films.

From the continuum point of view, a surface is regarded as a negligibly thin object adhering to the underlying material without slipping, and the material constants for both are different. A generic and mathematical exposition on surface elasticity has been presented by Gurtin and his coworkers (Gurtin and Murdoch, 1975; Gurtin et al., 1998). In their work, surface stress depends on deformation. The equilibrium and constitutive equations of the bulk solid are the same as those in the classical elasticity, but the boundary conditions must ensure the force balance of the surface object. This model has been applied by several authors. Besides Miller and Shenoy's work mentioned above, Shenoy (2002) analyzed size-dependent torsion of nanobars with prismatic section; Sharma and Ganti (2002) and Sharma et al. (2004) provided analytical expressions for the size-dependent strain states caused by spherical quantum dots and pores, respectively. Nevertheless, the presence of surface stress gives rise to a non-classical boundary condition which in combination with the constitutive relation of surface and the equations of classical elasticity forms a coupled system of field equations. This makes the solution of the corresponding boundary value problem relatively difficult. Many problems with more complicated geometry and loading condition remain to be solved, and the results are very important to the design of minute structural elements in a wide variety of nanomechanical systems.

The objective of this paper is to propose a general model for the mechanical analysis of ultra-thin elastic films of nano-scale thickness with arbitrary geometry and boundary condition. Upon the use of the transfer-matrix method and an expansion technique, it is shown that the mechanical response of the film is governed by three two-dimensional partial differential equations of the displacements on the mid-plane. The associated boundary conditions are described in an average manner in terms of the mid-plane displacements or force resultants, as is the case in practice. Therefore, solving the two-dimensional equations immediately gives the solution to the film. An intrinsic material parameter having length dimension appears naturally, implying that the solution is size-dependent. An illustrative example for axi-symmetric bending of a clamped circular film under the action of a concentrated force is given, and the result exhibits size-dependence consistent with the case in published literature. It is expected that the procedure can also be applied to study other problems of thin films, including both deformation and buckling.

2. Basic equations

The system under consideration is a flat elastic film with thickness h and arbitrarily shaped edge. The top and bottom surfaces of the film are free of external forces, but the edge can be subject to any boundary conditions. A rectangular coordinate system x_1, x_2 and x_3 is introduced, so that the mid-plane of the film coincides with the $x_1 - x_2$ plane. Throughout this paper, the usual summation convention is applied, where Latin subscripts run from 1 to 3 while Greek ones take the value of 1 or 2. A comma stands for differentiation with respect to the suffix coordinate.

Both the bulk and surfaces of the film are assumed elastically isotropic. The stress–strain relation of the bulk material is expressed by

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{1}$$

where σ_{ij} and u_i denote, respectively, stress and displacement, λ and μ are Lamé constants, and δ_{ij} is the Kronecker delta which is equal to 1 for $i = j$ and to 0 for $i \neq j$. In the absence of body force, the stress satisfies the static equilibrium equation

$$\sigma_{ij,j} = 0. \tag{2}$$

The constitutive relation of the surface proposed by Gurtin and Murdoch (1975) can be expressed as:

$$\tau_{\beta\alpha} = \tau^0\delta_{\beta\alpha} + (\lambda^s + \tau^0)u_{v,v}\delta_{\beta\alpha} + (\mu^s - \tau^0)(u_{\beta,\alpha} + u_{\alpha,\beta}), \tag{3}$$

in which $\tau_{\alpha\beta}$ is surface stress, τ^0 is residual surface tension under unconstrained conditions, and λ^s and μ^s are surface Lamé constants. These constants can be obtained by atomistic computation (Miller and Shenoy, 2000) and the change in residue surface tension can be determined from experimental data as shown in He and Lim (2001). Since the surfaces are flat, force balance of them requires that

$$\sigma_{\alpha 3} = \pm\tau_{\alpha\beta,\beta}, \quad \sigma_{33} = 0, \qquad \text{at } x_3 = \pm h/2. \tag{4}$$

Accordingly, for getting the solution to the film, one will confront with the field equations (1) and (2), along with the surface conditions defined by (3) and (4) as well as the edge boundary conditions to be specified.

In order to analyze the mechanical response of the film, the field equations (1) and (2) now are recast into the following transfer-matrix form:

$$\mathbf{p}_{,3} = \mathbf{A}\mathbf{q}, \quad \mathbf{q}_{,3} = \mathbf{B}\mathbf{p}, \quad \mathbf{r} = \mathbf{C}\mathbf{p}, \tag{5}$$

in which the 3×1 columns $\mathbf{p} \equiv \mathbf{p}(x_\alpha, x_3)$, $\mathbf{q} \equiv \mathbf{q}(x_\alpha, x_3)$, $\mathbf{r} \equiv \mathbf{r}(x_\alpha, x_3)$ and the 3×3 squares $\mathbf{A} \equiv \mathbf{A}(x_\alpha)$, $\mathbf{B} \equiv \mathbf{B}(x_\alpha)$ and $\mathbf{C} \equiv \mathbf{C}(x_\alpha)$ are defined by

$$\mathbf{p} = \begin{pmatrix} u_1 \\ u_2 \\ \sigma_{33} \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ u_3 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{11} & A_{23} \\ A_{13} & A_{23} & 0 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{13} \\ C_{31} & C_{32} & 0 \end{bmatrix}. \tag{6}$$

The elements of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are differential operators given by

$$\begin{aligned}
&A_{11} = \frac{1}{\mu}, \quad A_{13} = -(\ )_{,1}, \quad A_{23} = -(\ )_{,2};\\
&B_{11} = -\frac{4(1+\Lambda)\mu}{2+v}(\ )_{,11} - \mu(\ )_{,22}, \quad B_{12} = -\frac{2+3\Lambda}{2+\Lambda}\mu(\ )_{,12},\\
&B_{13} = -\frac{\Lambda}{2+\Lambda}(\ )_{,1}, \quad B_{22} = -\mu(\ )_{,11} - \frac{4(1+\Lambda)\mu}{2+\Lambda}(\ )_{,22},\\
&B_{23} = -\frac{\Lambda}{2+\Lambda}(\ )_{,2}, \quad B_{33} = -\frac{1}{(2+\Lambda)\mu};\\
&C_{11} = \frac{4(1+\Lambda)\mu}{2+\Lambda}(\ )_{,1}, \quad C_{12} = \frac{2\Lambda\mu}{2+\Lambda}(\ )_{,2},\\
&C_{13} = \frac{v}{2+\Lambda}, \quad C_{21} = \frac{2\Lambda\mu}{2+\Lambda}(\ )_{,1}, \quad C_{22} = \frac{4(1+\Lambda)\mu}{2+\Lambda}(\ )_{,2},\\
&C_{31} = \mu(\ )_{,2}, \quad C_{32} = \mu(\ )_{,1},
\end{aligned} \tag{7}$$

with $\Lambda = \lambda/\mu$. Note that the square matrices **A** and **B** are symmetric, but **C** is not. By substitution of Eq. (3), the surface boundary conditions in Eq. (4) in accordance with the notations defined in (6) are rewritten as

$$q_\alpha = \pm(\lambda^s + \mu^s)p_{v,v\alpha} \pm (\mu^s - \tau^0)\nabla^2 p_\alpha, \quad \text{at } x_3 = \pm h/2, \tag{8}$$

where $\nabla^2 = (\ )_{,vv}$ is the two-dimensional Laplacian.

3. Model formulation

The ratio of the thickness h to the in-plane characteristic dimension l of the film defines a small dimensionless parameter $\varepsilon = h/2l$. It is helpful to introduce a scaled thickness coordinate $z = x_3/\varepsilon$, referring to it the surfaces of the film are described by $z = \pm l$, and the first two equations in (5) are written as

$$\mathbf{p}_{,z} = \varepsilon\mathbf{A}\mathbf{q}, \quad \mathbf{q}_{,z} = \varepsilon\mathbf{B}\mathbf{p}. \tag{9}$$

The solution of the above equations can be assumed of the form

$$\mathbf{p} = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{p}^{(n)}, \quad \mathbf{q} = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{q}^{(n)}. \tag{10}$$

Substitution of this into Eqs. (9) and (8) leads to the equations

$$\mathbf{p}^{(0)}{}_{,z} = \mathbf{0}, \quad \mathbf{q}^{(0)}{}_{,z} = \mathbf{0}, \quad \mathbf{p}^{(n+1)}{}_{,z} = \mathbf{A}\mathbf{q}^{(n)}, \quad \mathbf{q}^{(n+1)}{}_{,z} = \mathbf{B}\mathbf{p}^{(n)}, \tag{11}$$

and the associated surface conditions

$$p_3^{(n)} = 0, \quad q_\alpha^{(n)} = \pm(1 + \Lambda^s)\mu^s p_{v,v\alpha}^{(n)} \pm (\mu^s - \tau^0)\nabla^2 p_\alpha^{(n)}, \quad \text{at } z = \pm l. \tag{12}$$

The first two equations in Eq. (11) imply that $\mathbf{p}^{(0)}$ and $\mathbf{q}^{(0)}$ are independent of z. Integrating them in combination with (12) yields

$$\mathbf{p}^{(0)} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ 0 \end{pmatrix}, \quad \mathbf{q}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ \bar{u}_3 \end{pmatrix}, \tag{13}$$

where $\bar{u}_\alpha$ and $\bar{u}_3$ are used to denote, respectively, $p_\alpha^{(0)}(x_\beta, 0)$ and $q_3^{(0)}(x_\beta, 0)$, for convenience. At the same time $\bar{u}_\alpha$ and $\bar{u}_3$ must satisfy

$$\frac{\lambda^s + 2\mu^s - \tau^0}{\mu^s - \tau^0}\Delta_{,1} + \Omega_{,2} = 0, \quad \frac{\lambda^s + 2\mu^s - \tau^0}{\mu^s - \tau^0}\Delta_{,2} - \Omega_{,1} = 0, \tag{14}$$

with Δ and Ω being defined by

$$\Delta = \bar{u}_{1,1} + \bar{u}_{2,2}, \quad \Omega = \bar{u}_{1,2} - \bar{u}_{2,1}. \tag{15}$$

In the following it will be seen that $\bar{u}_i$ are effectively the displacement components of a point on the mid-plane of the film. Applying $p_\alpha^{(1)}(x_\beta, 0) = q_3^{(1)}(x_\beta, 0) = 0$ and integrating the last two equations in (11) for $n = 0$ yield

$$\mathbf{p}^{(1)} = -z\begin{pmatrix} \bar{u}_{3,1} \\ \bar{u}_{3,2} \\ 0 \end{pmatrix}, \quad \mathbf{q}^{(1)} = -\mu\eta l\begin{pmatrix} (\nabla^2\bar{u}_3)_{,1} \\ (\nabla^2\bar{u}_3)_{,2} \\ 0 \end{pmatrix} - \frac{\Lambda}{2+\Lambda}z\begin{pmatrix} 0 \\ 0 \\ \Delta \end{pmatrix}, \tag{16}$$

where $\eta = (\lambda^s + 2\mu^s - \tau^0)/\mu$, and $\bar{u}_\alpha$ and $\bar{u}_3$ have to fulfill the equations

$$\frac{4(1+\Lambda)}{2+\Lambda}\Delta_{,1} + \Omega_{,2} = 0, \quad \frac{4(1+\Lambda)}{2+\Lambda}\Delta_{,2} - \Omega_{,1} = 0. \tag{17}$$

Note that η only depends on the material properties of the surface and bulk, hence defines an intrinsic length scale of the film.

The simultaneous satisfaction of both (14) and (17) demands that $\Delta_{,1} = \Delta_{,2} = \Omega_{,1} = \Omega_{,2} = 0$, namely

$$\Delta = \text{constant}, \quad \Omega = \text{constant}. \tag{18}$$

This result can be understood by considering a special case when the film undergoes an in-plane deformation. In this situation the in-plane displacement is $\bar{u}_\alpha$ across the whole thickness of the film, and the transverse shear stress $\sigma_{\alpha 3}$ vanishes. Equilibrium conditions for the surfaces and the bulk require that (14) and (17) must hold at the same time, resulting in constant Δ and Ω.

By using the results in (18), expressions for $\mathbf{p}^{(2)}$ and $\mathbf{q}^{(2)}$ can be obtained by integrating the last two in (11) for $n = 1$,

$$\begin{aligned} \mathbf{p}^{(2)} &= -\eta l z \begin{pmatrix} (\nabla^2\bar{u}_3)_{,1} \\ (\nabla^2\bar{u}_3)_{,2} \\ 0 \end{pmatrix}, \\ \mathbf{q}^{(2)} &= \frac{\Lambda}{2(2+\Lambda)}\mu \begin{pmatrix} 0 \\ 0 \\ \nabla^2\bar{u}_3 \end{pmatrix} + \frac{2(1+\Lambda)}{2+\Lambda}\mu(z^2 - l^2) \begin{pmatrix} (\nabla^2\bar{u}_3)_{,1} \\ (\nabla^2\bar{u}_3)_{,2} \\ 0 \end{pmatrix} \end{aligned} \tag{19}$$

and the following equation for $\bar{u}_3$ is derived as an additional result:

$$\nabla^2\nabla^2\bar{u}_3 = 0. \tag{20}$$

In passing these relations, $p_\alpha^{(2)}(x_\beta, 0) = q_3^{(2)}(x_\beta, 0) = 0$ are taken. Similarly, for $p_\alpha^{(3)}(x_\beta, 0) = q_3^{(3)}(x_\beta, 0) = 0$, continuing the above procedure for $n = 0$ yields

$$\mathbf{p}^{(3)} = \frac{4+3\Lambda}{6(2+\Lambda)}\left[z^2 - \frac{12(1+\Lambda)}{4+3\Lambda}l^2\right] z \begin{pmatrix} (\nabla^2\bar{u}_3)_{,1} \\ (\nabla^2\bar{u}_3)_{,2} \\ 0 \end{pmatrix}, \quad \mathbf{q}^{(3)} = \mathbf{0}. \tag{21}$$

It can be readily checked that the subsequent $\mathbf{p}^{(n)}$ and $\mathbf{q}^{(n)}$ $(n \geqslant 4)$ are all zero if $p_\alpha^{(n)}(x_\beta, 0) = q_3^{(n)}(x_\beta, 0) = 0$ are assumed before hand. Therefore, the expansions of $\mathbf{p}$ and $\mathbf{q}$ in (10) terminate after the fourth and third terms, respectively. Then the displacement field of the film is represented, referring to the original thickness coordinate x_3, by

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} - x_3 \begin{pmatrix} \bar{u}_{3,1} \\ \bar{u}_{3,2} \end{pmatrix} + f(x_3) \begin{pmatrix} (\nabla^2\bar{u}_3)_{,1} \\ (\nabla^2\bar{u}_3)_{,2} \end{pmatrix}, \\ u_3 &= \bar{u}_3 - \frac{\Lambda}{2+\Lambda}x_3\Delta + \frac{\Lambda}{2(2+\Lambda)}x_3^2\nabla^2\bar{u}_3, \end{aligned} \tag{22}$$

and the stress field in the bulk material reads

$$\begin{aligned}
&\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \end{pmatrix} = \frac{2\mu}{2+\Lambda}\begin{bmatrix} 2(1+\Lambda) & \Lambda \\ \Lambda & 2(1+\Lambda) \end{bmatrix}\left[\begin{pmatrix} \bar{u}_{1,1} \\ \bar{u}_{2,2} \end{pmatrix} - x_3\begin{pmatrix} \bar{u}_{3,11} \\ \bar{u}_{3,22} \end{pmatrix} + f(x_3)\begin{pmatrix} (\nabla^2\bar{u}_3)_{,11} \\ (\nabla^2\bar{u}_3)_{,22} \end{pmatrix}\right], \\
&\sigma_{12} = \mu\left[\bar{u}_{1,2} + \bar{u}_{2,1} - 2x_3\bar{u}_{3,12} + 2f(x_3)(\nabla^2\bar{u}_3)_{,12}\right], \\
&\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \mu g(x_3)\begin{pmatrix} (\nabla^2\bar{u}_3)_{,1} \\ (\nabla^2\bar{u}_3)_{,2} \end{pmatrix}, \quad \sigma_{33} = 0.
\end{aligned} \tag{23}$$

Here the functions $f(x_3)$ and $g(x_3)$ are defined by

$$\begin{aligned}
f(x_3) &= \frac{1}{2}\left\{\frac{4+3\Lambda}{3(2+\Lambda)}\left[x_3^2 - \frac{3(1+\Lambda)}{4+3\Lambda}h^2\right] - \eta h\right\}x_3, \\
g(x_3) &= \frac{1}{2}\left[\frac{1+\Lambda}{2+\Lambda}(4x_3^2 - h^2) - \eta h\right].
\end{aligned} \tag{24}$$

Although the results in (22) and (23) are not the most general solution to the equations in (5), they are the asymptotic solutions within the scope of this paper. Although the approach is asymptotic, the solutions can be regarded as exact solutions because the expansion terminates after the third or the fourth terms as detailed above. Eqs. (18) and (20) are mathematically similar to the governing equations in the classical plate theory (Timoshenko and Woinowsky-Krieger, 1959). The associated boundary conditions at the edge of the film can be specified in an average manner in terms of mid-plane displacements or rotations, stress resultants and moments. Indeed, this is consistent with the real case in practice where a point-by-point prescription of boundary condition across the thickness of a thin film is unrealistic. Once $\bar{u}_i$ are solved from (18) and (20), the mechanical field in the film is generated from (22) and (23), and an intrinsic length scale arises naturally. According to Saint Venant's principle (Timoshenko and Gere, 1951), the solution is exact in the interior of the film far from the edge with a distance of order of film thickness.

4. Illustrative example

A practical example of a clamped circular film of radius R is presented to illustrate the exact solution procedure using the present model. The film is subject to a concentrated force P that is caused by the action of a tip along the x_3-direction as shown in Fig. 1. In this case, it is convenient to introduce a cylindrical coordinate system r, θ and x_3 where the origin is located at the center of the film, and r and θ is related to x_1 and x_2 by $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Due to the symmetry of the problem, one has $\bar{u}_r = \bar{u}_r(r)$, $\bar{u}_\theta = 0$ and $\bar{u}_3 = \bar{u}_3(r)$.

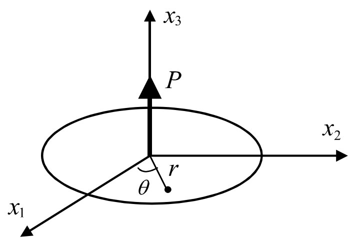

Fig. 1. A clamped circular film subject to a concentrated force P at the center.

The governing equations for the problem read

$$\frac{1}{r}(r\bar{u}_r)_{,r} = c, \quad \left\{ r\left[\frac{1}{r}(r\bar{u}_{3,r})_{,r}\right]_{,r} \right\}_{,r} = 0, \tag{25}$$

with c being a constant to be determined. The associated boundary conditions are

$$\bar{u}_r = 0, \qquad \bar{u}_3 = 0, \qquad \bar{u}_{3,r} = 0, \quad \text{at } r = R. \tag{26}$$

Other conditions include the finite values of $\bar{u}_r$ and $\bar{u}_3$ at $r = 0$ and the equilibrium of the force P by stresses in the film. The latter can be derived as follows. Imagine that a small axisymmetric element with radius r with respect to the x_3-axis is cut from the film. Mechanical equilibrium of the small axisymmetric element requires that

$$\int_{-h/2}^{h/2} \sigma_{r3}\, \mathrm{d}x_3 = -\frac{P}{2\pi r}. \tag{27}$$

Under these conditions, the solution of (25) can be obtained as

$$\bar{u}_1 = 0, \quad \bar{u}_3 = w\left[1 - \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right)^2 \ln\frac{r}{R}\right], \tag{28}$$

where

$$w = \frac{3(2+\Lambda)R^2P}{16\pi\mu h^2[(1+\Lambda)h + 6(2+\Lambda)\eta]}, \tag{29}$$

is the central deflection of the film. The displacement and stress fields of the film then are expressed by

$$\begin{aligned}
u_r &= \frac{4w}{R^2 r}\left[\frac{4+3\Lambda}{3(2+\Lambda)}x_3^2 - \left(\eta + \frac{1+\Lambda}{2+\Lambda}h\right)h - r^2 \ln\frac{r}{R}\right]x_3, \\
u_3 &= \frac{w}{R^2}\left[\frac{2\Lambda}{2+\Lambda}\left(1 + 2\ln\frac{r}{R}\right)x_3^2 - r^2\left(1 - 2\ln\frac{r}{R}\right) + R^2\right], \\
\sigma_{rr} &= -\frac{\mu w}{R^2 r^2}\left[\frac{4+3\Lambda}{3(2+\Lambda)}x_3^2 - \left(\eta + \frac{1+\Lambda}{2+\Lambda}h\right)h + \frac{1+\Lambda}{2+\Lambda}r^2\left(2 + \frac{2+3\Lambda}{1+\Lambda}\ln\frac{r}{R}\right)\right]x_3, \\
\sigma_{r3} &= \frac{4\mu w}{R^2 r}\left[\frac{4(1+\Lambda)}{2+\Lambda}x_3^2 - \left(\eta + \frac{1+\Lambda}{2+\Lambda}h\right)h\right].
\end{aligned} \tag{30}$$

Note that the concentrated force gives rise to singular stresses at $r = 0$.

Apparently, the above solution contains the intrinsic length η and, hence, is size-dependent. It is clear that the size-dependent nature of the mechanical response of the film originates from strain-dependence of surface stress, as is described in Eq. (3). Indeed, if the surface stress is assumed constant, corresponding to the limit case of $\eta \to 0$, the results in (30) will be scaling-invariant. In particular, the central deflection of the film, now denoted by w^0, becomes

$$w^0 = \frac{3(2+\Lambda)R^2P}{16\pi\mu(1+\Lambda)h^3}. \tag{31}$$

This is exactly the same as the prediction of classical elasticity without surface effect. To illustrate the size-dependence quantitatively, the difference between w and w^0 will be examined. Miller and Shenoy (2000) have computed free-surface properties for aluminum and silicon, and indicated different values depending upon crystallographic orientation. The intrinsic length they defined (the modulus ratio of surface to bulk) can be either positive or negative, but the absolute value is nearly 1 Å. According to their data, it can be calculated that the relevant parameters in the present model are basically of the same orders. Fig. 2 shows

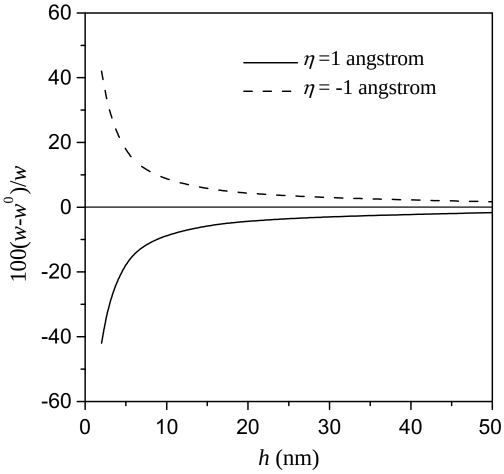

Fig. 2. Relative error of the central deflection as a function of film thickness.

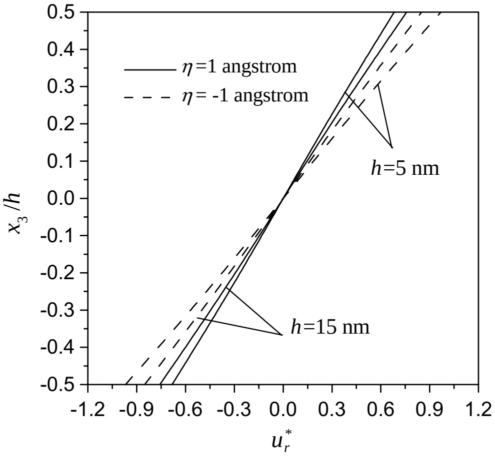

Fig. 3. Through-the-thickness distribution of the dimensionless in-plane displacement at $r = R/2$.

the relative errors between w and w^0, $(w - w^0)/w$, for various values of film thickness, where $\Lambda = 1.5$ and $\eta = 1$ or -1 Å are taken. It can be seen that, positive value of η makes the film stiffer while negative value of η makes the film more compliant. Because η depends on the surface Lamé constants λ^s and μ^s and the residual surface tension under unconstrained conditions τ^0 as defined in Eq. (16), it implies larger values of λ^s and μ^s increase the stiffness of the film while excessive τ^0 reduces its stiffness. For both cases the relative error of the central deflection increases with the decrease of the film thickness, and the size-dependence becomes significant when the film thickness falls down to about 20 nm. The results are in reasonable consistency with the other predictions by lattice model (Sun and Zhang, 2003) or continuum method (Gurtin and Murdoch, 1975; Shenoy, 2002; Sharma and Ganti, 2002; Sharma et al., 2003).

Due to the effect of surfaces, the distributions of displacements and stresses within the film also depend on the thickness. For elucidation, the following dimensionless quantities are introduced

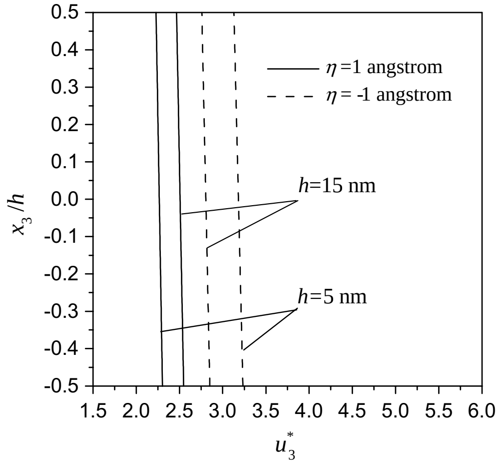

Fig. 4. Through-the-thickness distribution of the dimensionless out-of-plane displacement at $r = R/2$.

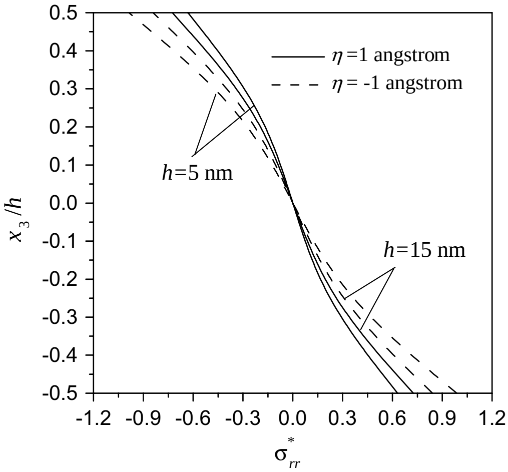

Fig. 5. Through-the-thickness distribution of the dimensionless in-plane stress at $r = R/2$.

$$u_r^* = \frac{\pi h \mu}{P} u_r, \quad u_3^* = \frac{\pi h \mu}{P} u_3, \quad \sigma_{rr}^* = \frac{\pi h^2}{P} \sigma_{rr}, \quad \sigma_{r3}^* = \frac{\pi h^2}{P} \sigma_{r3}. \tag{32}$$

When the surface effects are excluded, i.e. $\eta \to 0$, it can be known from (30) that the relationships between these quantities and x_3/h are scaling-invariant or size-independent. However, this is no longer the case for ultra-thin films of nano-scale thickness where the surface effects cannot be ignored. Evident numerical results for the distributions of the displacements and stresses at $r = R/2$ are given in the following, where $R/h = 5$ is taken and the cases for $h = 5$ nm and 15 nm are compared.

Plotted in Figs. 3 and 4 are through-the-thickness distributions of in-plane and out-of-plane displacements, u_r^* and u_3^*. Since positive η increases the effective stiffness while negative η decreases it, the magnitudes of the dimensionless displacements for $\eta = -1$ Å are greater than those for $\eta = 1$ Å. When the film thickness is decreased, the absolute values of the dimensionless displacements decrease for positive η while

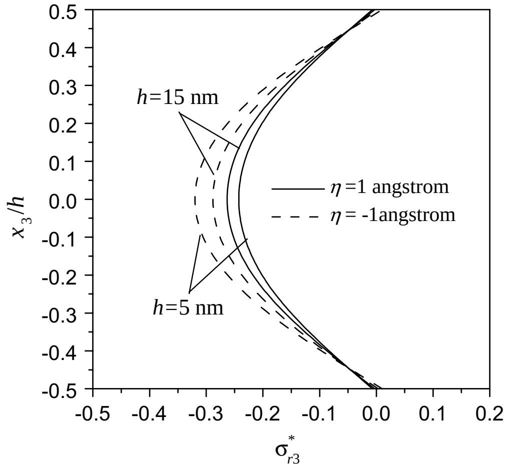

Fig. 6. Through-the-thickness distribution of the dimensionless transverse shear stress at $r = R/2$.

increases for negative η. In both cases the variations of the displacements are nearly linear across the thickness. In the classical theory for elastic plates without surface effects, a linearly varying in-plane displacement and a constant out-of-plane displacement are assumed. The corresponding transverse shear stresses then vanish. Nevertheless, for films with surface effects, these simplifications will lead to inconsistency of stresses with respect to the boundary conditions on the surface.

The distributions of the dimensionless in-plane and transverse shear stress, σ^*_{rr} and σ^*_{r3}, through the thickness of the bulk material at $r = R/2$ are shown in Figs. 5 and 6, respectively. Again, the magnitudes of these stresses are greater for $\eta = -1$ Å than for $\eta = 1$ Å at the same position. The effects of variation in film thickness on σ^*_{rr} and σ^*_{r3} are also similar to the effects on displacements. Note that the maximum transverse shear stress is greater than the maximum in-plane stress in magnitude, because for the problem under consideration the external load is mainly balanced by the resultant shear force in the film. The transverse shear stresses in the regions very close to the surfaces, though quite small, do not vanish. They can be positive or negative, depending on the sign of η.

5. Conclusions

A continuum model based on surface elasticity is proposed to analyze the size-dependent mechanical response of ultra-thin elastic films of nano-scale thickness. Being expressed in terms of displacements of the mid-plane, the governing equations are two-dimensional and the associated boundary conditions are specified at the edge of the film in an average manner as in the classical plate theory. Once the two-dimensional equations are solved, the three-dimensional mechanical field that is exact in Saint Venant's sense is generated directly. The asymptotic analysis developed for solving the two-dimensional equations can be regarded to yield exact solutions because the expansion terminates after a few terms. The solution procedure is illustrated by analyzing a clamped circular film under a concentrated force. The result is consistent with the other existing studies and it approaches the classical plate solution without surface stress effects. It is concluded that the size-dependence is due to the dependence of surface stress on strain. Ignoring this strain-dependence of surface stress will lead to the disappearance of size effect. The presence of surface Lamé constants and residual surface tension under unconstrained conditions increases and decreases the film stiffness, respectively.

Acknowledgements

The work described in this paper was fully supported by grants from the Research Grant Council of the Hong Kong Special Administrative Region [Project No. CityU 1036/01E] and City University of Hong Kong [Project No. 7001475 (BC)].

References

Gurtin, M.E., Murdoch, A., 1975. A continuum theory of elastic material surfaces. Archives of Rational Mechanics and Analysis 57, 291–323.

Gurtin, M.E., Weissmuller, J., Larche, F., 1998. A general theory of curved deformable interfaces in solids at equilibrium. Philosophical Magazine A 78, 1093–1109.

He, L.H., Lim, C.W., 2001. On the bending of unconstrained thin crystalline plates caused by change in surface stress. Surface Science 478, 203–210.

Miller, R.E., Shenoy, V.B., 2000. Size-dependent elastic properties of nanosized structural elements. Nanotechnology 11, 139–147.

Rose, K.C., Wang, J., Hutchinson J.W., Lieber, C.M., 2000. Nanoplate mechanics: dramatic decrease in Young's modulus, unpublished.

Sharma, P., Ganti, S., 2002. Interfacial elasticity corrections to size-dependent strain-state of embedded quantum dots. Physique Status Solida B 234, R10–R12.

Sharma, P., Ganti, S., Bhate, N., 2003. Effect of surfaces on the size-dependent elastic state of nano-inhomogeneities. Applied Physics Letters 82, 535–537.

Sheehan, P.E., Lieber, C.M., 1996. Nanotribology and nanofabrication of Moo_3 structures by atomic force microscopy. Science 272, 1156–1161.

Shenoy, V.B., 2002. Size-dependent rigidities of nanosized torsional elements. International Journal of Solids and Structures 39, 4039–4052.

Sun, C.T., Zhang, H., 2003. Size-dependent elastic moduli of platelike nanomaterials. Journal of Applied Physics 93, 1212–1218.

Terrones, M., Grobert, N., Hsu, W., Hu, Y., Terrones, H., Hare, J., Kroto, H., Walton, D., 1999. Advances in the creation of filled nanotubes and novel nanowires. Material Research Society Bulletin 24, 43–49.

Timoshenko, S., Gere, J.M., 1951. Theory of Elasticity. McGraw-Hill, New York.

Timoshenko, S., Woinowsky-Krieger, S., 1959. Theory of Plates and Shells. McGraw-Hill, New York.

Wong, E., Sheehan, P.E., Lieber, C.M., 1997. Nanobeam mechanics: elasticity, strength, and toughness of nanorod sand nanotubes. Science 277, 1971–1975.

Yakobson, B.I., Smalley, R., 1997. Fullerene nanotubes: $C_{1,000,000}$and beyond. American Science 85, 324–337.